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Analysis of Log-Weighted Quadrature Domains
and the Faber Transform

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Caltech Department of Mathematics

UT Dallas Complex (Analysis and Dynamics) Conference II
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Mean value property:[1]

$$f \in \mathcal{A}(\mathbb{D}_r(w_0)) \quad \Longrightarrow \quad \frac{1}{r^2}\int_{\mathbb{D}_r(w_0)} f dA = f(w_0).$$

Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

[1] $dA = \frac{dxdy}{\pi}$
[2] bounded & simply connected

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The cardioid, $\Omega = \left\{ z + \frac{z^2}{2} : z \in \mathbb{D} \right\}$:

$$f \in \mathcal{A}(\Omega) \quad \Longrightarrow \quad \int_\Omega f dA = \frac{3}{2} f(0) + \frac{1}{2} f'(0).$$

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Aharonov & Shapiro (1976): The cardioid is also unique.

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These are examples of *quadrature identities*.

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Bounded Quadrature Domains

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Definition 1.1 (Bounded Quadrature domain)

A bounded domain $\Omega \subset \widehat{\mathbb{C}}$ is a *bounded* QD if there exists rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

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This is equivalent to the existence of a quadrature identity:

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{p_k \in h^{-1}(\infty)} \operatorname*{Res}_{w=p_k} (f(w)h(w)) = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k).$$

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Definition 1.2 (Unbounded Quadrature Domain)

An unbounded domain $\Omega \subset \widehat{\mathbb{C}}$ is an *unbounded* QD if $\exists$ a rational h s.t.

$$\int_{\Omega} f dA = \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}_0(\Omega)$. This is denoted by $\Omega \in \mathrm{QD}(h)$.

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unbounded QD $\longleftrightarrow$ quadrature identity

$$\oint_{\partial\Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) + \sum_j c_j f_j$$

where $f(w) = \sum_{j=1}^{\infty} f_j w^{-j}$.

Characterizations of QDs I: "Potato Kugel"[3]

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Definition 1.3 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^\Omega : \mathbb{C} \to \mathbb{C}$,

$$C^\Omega(w) = \int_\Omega \frac{dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}(h) \iff C^\Omega = h \text{ in } \Omega_*$$

[3] Aharonov, Schiffer & Zalcman (1981)

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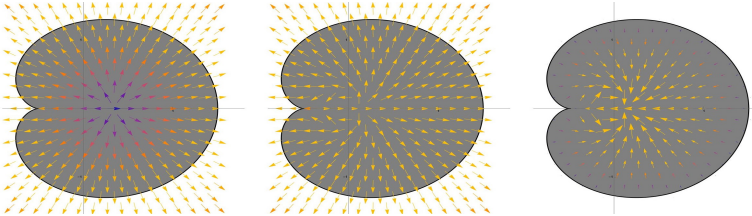
$\overline{C^\Omega}$ corresponds to the electric field due to a uniform charge distribution on Ω.



$\overline{C^\Omega}$ $\qquad$ $\overline{h}$ $\qquad$ $\overline{C^\Omega} - \overline{h}$

[3] Aharonov, Schiffer & Zalcman (1981)

Characterizations of QDs II: The Coincidence Equation

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$\Omega \subset \widehat{\mathbb{C}}$ is a QD iff there exists a rational function h such that

$$\overline{w} = h(w) + C^{\Omega^c}(w), \quad \forall w \in \partial\Omega.$$

Referred to as the *coincidence equation*.

Characterizations of QDs II: The Coincidence Equation

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$$\overline{w} = h(w) + C^{\Omega^c}(w), \quad \forall w \in \partial\Omega.$$

Referred to as the *coincidence equation*.

We interpret this identity in terms of logarithmic potential theory:
If $K = \Omega^c$ is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}(H(w)),$$

then

$$\frac{\partial}{\partial \overline{w}}\left(Q(w) + U^{\mu}(w)\right) = 0, \quad \forall w \in K.$$

If we set $h = H'$ then differentiating, yields the coincidence equation.

Characterizations of QDs III: The Schwarz Function

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$\Omega \subset \widehat{\mathbb{C}}$ is a QD $\iff$ Ω admits a *Schwarz function*.

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$$\Omega \subset \widehat{\mathbb{C}} \text{ is a QD} \iff \Omega \text{ admits a } \textit{Schwarz function}.$$

S is called a *Schwarz function* for Ω if:

1. $S \in \mathcal{M}(\Omega)$,
2. S extends continuously to $\partial\Omega$,
3. $S(w) = \overline{w}$ on $\partial\Omega$.

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S is called a *Schwarz function* for Ω if:

1. $S \in \mathcal{M}(\Omega)$,
2. S extends continuously to $\partial\Omega$,
3. $S(w) = \overline{w}$ on $\partial\Omega$.

S can be represented explicitly in terms of h and the Cauchy transform:

$$S(w) = h(w) + C^{\Omega^c}(w).$$

Characterizations of QDs IV: The Riemann Map

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If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

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If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

Moreover, if $\Omega \in \mathrm{QD}(h)$ is a **bounded** domain with Riemann map φ, then

$$h(w) = \Phi_\varphi\left(\varphi^\# - \overline{\varphi(0)}\right)(w)$$

$$\varphi(z) = \varphi(0) + \Phi_\varphi^{-1}(h)^\#(z)$$

Characterizations of QDs IV: The Riemann Map

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If Ω is simply connected, then $\Omega \in \mathrm{QD}$ iff its Riemann map is rational.

Moreover, if $\Omega \in \mathrm{QD}(h)$ is a **bounded** domain with Riemann map φ, then

$$h(w) = \Phi_\varphi\left(\varphi^\# - \overline{\varphi(0)}\right)(w)$$

$$\varphi(z) = \varphi(0) + \Phi_\varphi^{-1}(h)^\#(z)$$

If Ω is **unbounded**, then

$$h(w) = \Phi_\varphi\left(\varphi^\# - cz^{-1}\right)(w)$$

$$\varphi(z) = cz + \Phi_\varphi^{-1}(h)^\#(z)$$

Where Φ_φ is the *Faber transform*.

Summary of QD Equivalent Characterizations

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The following are equivalent to $\Omega \in \mathrm{QD}(h)$:

1. $C^{\Omega} = h$ in Ω_*;
2. Ω satisfies the coincidence equation;
3. Ω admits a Schwarz function;
4. Ω has a rational Riemann map (when simply connected).

The Faber Transform

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Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}_* \to \Omega$,

$$\varphi(z) = cz + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots, \qquad c = \mathrm{rad}_\infty(\Omega) > 0$$

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The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega_*)$,

$$\Phi_\varphi(f)(w) = \oint_{\partial \mathbb{D}_*} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz$$

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Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

$$\Phi_\varphi\left(z^n\right)(w) =: F_n(w) \quad (n\text{th Faber polynomial})$$

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The *interior* Faber transform, defined for bounded s.c. domains is defined similarly, and is a map $\Phi_\varphi : \mathcal{A}_0(\mathbb{D}_*) \to \mathcal{A}_0(\Omega_*)$.

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Definition 2.1 (Log-Weighted Quadrature Domain)

Let $\Omega \subset \widehat{\mathbb{C}}$ be a domain for which $0, \infty \notin \partial\Omega$. We say that Ω is a log-weighted quadrature domain (LQD) if there exists a rational h such that

$$\int_{\Omega} f(w)|w|^{-2} dA(w) = \oint_{\partial\Omega} f(w)h(w)dw, \qquad \forall f \in L^1_a(\Omega; \rho_0).$$

Denoted by $\Omega \in \mathrm{QD}_0(h)$.

That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

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That is, LQDs are quadrature domains with respect to the weight $\rho_0(w) = |w|^{-2}$.

- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$.

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- We refer to LQDs containing zero as *singular*.

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- The singularity plays a similar role to $\infty \in \widehat{\mathbb{C}}$ for classical QDs.

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- The primary distinction of LQDs from classical QDs is the metric singularity at $w = 0$, reflected in the constraint $f \in L^1_a(\Omega; \rho_0) \Rightarrow f(0) = 0$.
- We refer to LQDs containing zero as *singular*.
- The singularity plays a similar role to $\infty \in \widehat{\mathbb{C}}$ for classical QDs.
- h is *not* always unique: If $0 \in \Omega \in \mathrm{QD}_0(h)$, then $\Omega \in \mathrm{QD}_0\left(h(w) + \frac{q}{w}\right)$ for all $q \in \mathbb{C}$.

A First Example

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Fact: If Ω is a bounded domain, then

$$\Omega \in \mathrm{QD}(h) \quad \Longrightarrow \quad e^{\Omega} \in \mathrm{QD}_0(\widetilde{h}),$$

for some rational $\widetilde{h}$. Moreover, if $\{p_k\}$ are the poles of h then $\{e^{p_k}\}$ are the poles of $\widetilde{h}$, with the same multiplicity.

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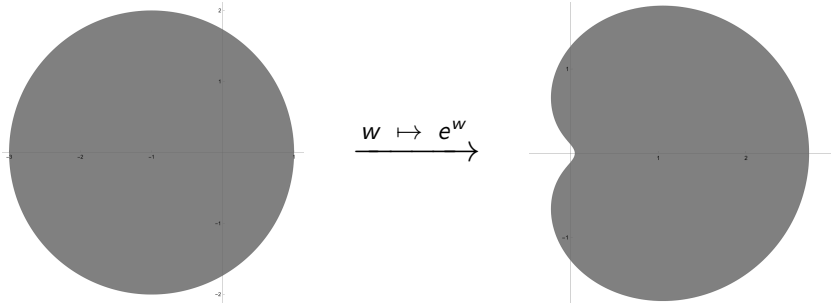
for some rational $\widetilde{h}$. Moreover, if $\{p_k\}$ are the poles of h then $\{e^{p_k}\}$ are the poles of $\widetilde{h}$, with the same multiplicity.

$$\mathbb{D}_2(-1) \in \mathrm{QD}\left(\frac{4}{w+1}\right) \quad \Longrightarrow \quad e^{\mathbb{D}_2(-1)} \in \mathrm{QD}_0\left(\frac{4}{w-e^{-1}}\right)$$



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For a Borel set $\Omega \subset \mathbb{C}$, we denote the ρ_0-*weighted Cauchy transform* of Ω by $C^{\Omega}_{\rho_0} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}_{\rho_0}(w) = \int_{\Omega} \frac{|\xi|^{-2} dA(\xi)}{w - \xi}$$

$$\Omega \in \mathrm{QD}_0 \iff C^{\Omega}_{\rho_0} \text{ extends to a rational function.}$$

$\overline{C^{\Omega}_{\rho_0}}$ corresponds to the electric field due to a charge density $\rho_0(w) = |w|^{-2}$ on Ω:

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Definition 2.2 (Weighted Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the ρ_0-*weighted Cauchy transform* of Ω by $C^{\Omega}_{\rho_0} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}_{\rho_0}(w) = \int_{\Omega} \frac{|\xi|^{-2} dA(\xi)}{w - \xi}$$

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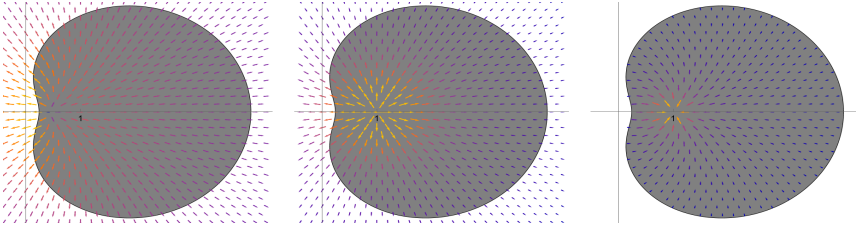

$\overline{C^{\Omega}_{\rho_0}}$ $\qquad$ $\overline{h}$ $\qquad$ $\overline{C^{\Omega}_{\rho_0}} - \overline{h}$

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Non-singular coincidence equation: A domain $0 \notin \Omega$ is an LQD iff there exist $G \in \mathcal{A}(\Omega)$ and a rational function h such that

$$\frac{\ln |w|^2}{w} = h(w) + G(w), \quad \forall w \in \partial\Omega.$$

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$$\frac{\ln |w|^2}{w} = h(w) + G(w), \quad \forall w \in \partial\Omega.$$

Singular coincidence equation: A domain $0 \in \Omega$ is an LQD iff there exist $q \in \mathbb{C}$, $G \in \mathcal{A}(\Omega)$, and a rational function h such that

$$\frac{\ln |w|^2}{w} = h(w) + \frac{q}{w} + G(w), \quad \forall w \in \partial\Omega.$$

In both cases, $\Omega \in \mathrm{QD}_0(h)$.

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Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

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Theorem 2.3

A domain Ω is an LQD iff it admits a generalized Schwarz function.

S_0 is called a *generalized Schwarz function* for Ω if

1. $S_0 \in \mathcal{M}(\Omega)$,
2. S_0 extends continuously to $\partial\Omega$,
3. $S_0(w) = \dfrac{\ln|w|^2}{w}$ on $\partial\Omega$.

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Remark: Solving for $\overline{w}$, we recover an analogue of the classical Schwarz function $S|_{\partial\Omega}(w) = \overline{w}$:

$$S(w) := \frac{1}{w} e^{wS_0(w)}.$$

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Not typically a Schwarz function (essential singularity $\implies$ not meromorphic).

Characterizations of LQDs IV: The Riemann Map

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Let Ω be a simply connected domain with Riemann map φ.

Write the *inner/outer factorization* of φ:

$$\varphi(z) = \varphi_{\mathrm{in}}(z)\varphi_{\mathrm{out}}(z).$$

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Theorem 2.4

$\Omega \in QD_0$ iff $\ln(\varphi_{\mathrm{out}})$ extends to a rational function.

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$\Omega \in QD_0$ iff $\ln(\varphi_{\text{out}})$ extends to a rational function.

$\varphi(z) = \varphi_{\text{in}}(z)\varphi_{\text{out}}(z)$ in the sense of Nevanlinna theory.

In general:

- φ_{in} is a ratio of Blaschke factors:

$$b_{z_0}(z) := \frac{\overline{z_0}}{|z_0|}\frac{z - z_0}{\overline{z_0}z - 1},$$

- φ_{out} is non-zero and analytic.

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If Ω is a *bounded* domain, then we can write the inner/outer factorization of φ as follows:

$$\varphi(z) = 1 \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \notin \Omega,$$
$$\varphi(z) = b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \in \Omega,$$

where z_0 is the unique root of φ in $\mathbb{D}$.

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where z_0 is the unique root of φ in $\mathbb{D}$.

If Ω is an *unbounded* domain, then

$$\varphi(z) = z \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \notin \Omega,$$
$$\varphi(z) = z b_{z_0}(z) \cdot \varphi_{\mathrm{out}}(z), \qquad \text{when } 0 \in \Omega,$$

where z_0 is the unique root of φ in $\mathbb{D}_*$.

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where z_0 is the unique root of φ in $\mathbb{D}_*$.

Note: This factorization is key to relating the Riemann map and quadrature function.

Summary of LQD Equivalent Characterizations

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The following are equivalent to $\Omega \in \mathrm{QD}_0(h)$:

1. The weighted Cauchy transform:
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega} = h$,
   - $0 \notin \Omega$: $C_{\rho_0}^{\Omega_0 \setminus \mathbb{D}_r}(w) = \dfrac{q}{w} + h(w)$.

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$$S_0(w) = \frac{\ln |w|^2}{w}, \quad \forall w \in \partial\Omega.$$
4. $\ln(\varphi_{\mathrm{out}})$ extends to a rational function (if simply connected).

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Theorem 2.5 (Boundary regularity)

If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

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Theorem 2.5 (Boundary regularity)

If $\Omega \in QD_0$ then $\partial\Omega$ has finitely many singular points, and each is a cusp or a double point.

Theorem 2.6 (Scaling law)

For each $a \in \mathbb{C} \setminus \{0\}$, $\Omega \in QD_0(h)$ if and only if $a^{-1}\Omega \in QD_0(h(aw)a)$.

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Theorem 2.7 (Invariance under inversion)

If $\Omega \subseteq \widehat{\mathbb{C}}$ then $\Omega \in QD_0(h)$ iff $\Omega^{-1} \in QD_0(-h(w^{-1})w^{-2})$.

(the singularities at $w = 0$ and $w = \infty$ are "interchangable")

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Question: When is an LQD uniquely associated to its quadrature function? How can we recover it?

These are open problems even for *classical* QDs.

[4] Novikoff, P.P.: Sur le problème inverse du potentiel. (1938)

[5] Gustafsson, B., Putinar, M.: Selected topics on quadrature domains. (2007)

[6] Aharonov, D., Shapiro, H.S.: Domains on which analytic functions satisfy quadrature identities. (1976)

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- If a s.c. bounded domain $\Omega \in \mathrm{QD}(h)$ for some $h(w) = \sum_{j=1}^{n} \frac{c_j}{w - w_j}$ with the $c_j > 0$ and $\mathrm{diam}\left(\{w_j\}\right) < \sqrt{c_1 + \ldots + c_n}$, then Ω is unique.[5]

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- If a s.c. bounded domain $\Omega \in \mathrm{QD}\left(\frac{c_1}{w - w_0} + \frac{c_2}{(w - w_0)^2}\right)$ for some $c_1 > 0$ and $c_2 \in \mathbb{C}$, then Ω is unique.[6]

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Uniqueness of Null LQDs

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A domain Ω is referred to as a *null LQD* if $\Omega \in \mathrm{QD}_0(0)$. We prove the following classification theorem for null LQDs.

Theorem 2.8

A domain is a null LQD if and only if it is a disk or exterior disk centered at 0.

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Recall that a domain is a null QD iff it is an exterior disk.

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Question: Why are disks null LQDs but not null classical QDs?

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Recall that a domain is a null QD iff it is an exterior disk.

Question: Why are disks null LQDs but not null classical QDs?

Answer: If $\Omega \in \mathrm{QD}_0(0)$, then Ω must contain either 0 or ∞ because otherwise $1 \in L^1_a(\Omega; \rho_0)$, so

$$0 < \int_\Omega |w|^{-2} dA(w) = \oint_{\partial\Omega} 1 \cdot 0 dw = 0.$$

Whereas, for classical QDs $\Omega \in \mathrm{QD}_0(0)$ implies only $\infty \in \Omega$.

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Theorem 2.9

Let $\Omega \in QD_0(h)$ be a bounded simply connected domain with Riemann map φ. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that[7]

$$\begin{aligned} \varphi(z) &= w_0 e^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{1}$$

Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$.

[7] $r^{\#}(z) := \overline{r(\overline{z^{-1}})}$.

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Theorem 2.9

Let $\Omega \in QD_0(h)$ be a bounded simply connected domain with Riemann map φ. Then there exists rational $r \in \mathcal{A}_0(\mathbb{D}_*)$ such that[7]

$$\begin{aligned} \varphi(z) &= w_0 e^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= \frac{w_0}{|z_0|} b_{z_0}(z) e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{1}$$

Where $\varphi(0) = w_0$ and $\varphi(z_0) = 0$. In this case,

$$r(z) = \Phi_\varphi^{-1}\left(wh(w) + \underset{\infty}{\mathrm{Res}}\, h\right)(z), \tag{2}$$

and

$$h(w) = \frac{\Phi_\varphi(r)(w) + C}{w} \tag{3}$$

[7] $r^{\#}(z) := \overline{r(\overline{z^{-1}})}$.

Example: Non-Singular Bounded One Point LQDs

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Theorem 2.10

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

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$\implies$ **"Proof":** Suppose $\Omega \in \mathrm{QD}_0\left(\frac{\alpha}{w-w_0}\right)$ with Riemann map φ s.t. $\varphi(0) = w_0$. By the theorem,

$$\varphi(z) = w_0 e^{r^{\#}(z)},$$

where

$$r(z) = \alpha w_0 \Phi_\varphi^{-1}\left(\frac{1}{w-w_0}\right)(z) = \alpha w_0 \frac{\psi'(w_0)}{z - \psi(w_0)} = \frac{\alpha w_0}{\varphi'(0)} \frac{1}{z}$$

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Hence, $\varphi(z) = w_0 e^{\frac{\alpha w_0}{\varphi'(0)} z}$. Solving for $\varphi'(0)$, we recover

$$\varphi(z) = w_0 e^{z\sqrt{\alpha}},$$

which is univalent in $\mathbb{D}$ iff $0 < \alpha \leq \pi^2$.

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Theorem 2.11

Fix $\alpha > 0$ and $w_0 \in \mathbb{C} \setminus \{0\}$. If $0 \notin \Omega$ is simply connected and bounded, then $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ iff $0 < \alpha \leq \pi^2$ and $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$.

$\Longleftarrow$ **"Proof":** Suppose $\Omega = \varphi(\mathbb{D})$, where $\varphi(z) = w_0 e^{z\sqrt{\alpha}}$. $\ln(\varphi_{\text{out}})$ is rational, so $\Omega \in \mathrm{QD}_0(h)$.

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Then, by the theorem,

$$h(w) - \frac{C}{w} = \frac{\sqrt{\alpha}}{w}\Phi_\varphi\left(z^{-1}\right)(w) = \frac{\sqrt{\alpha}}{w}\frac{\varphi'(0)}{w-\varphi(0)} = \frac{\alpha}{w}\frac{w_0}{w-w_0}$$

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Applying partial fraction decomposition, we obtain

$$h(w) = \frac{\alpha}{w-w_0} + \frac{C-\alpha}{w}.$$

Given that $0 \notin \Omega$, we find that $C - \alpha = 0$, and the conclusion follows.

Classification of One Point LQDs

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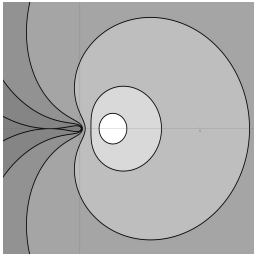
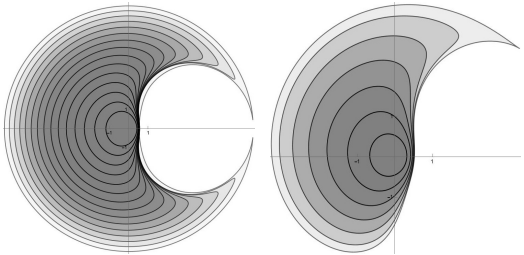
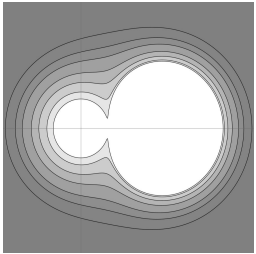
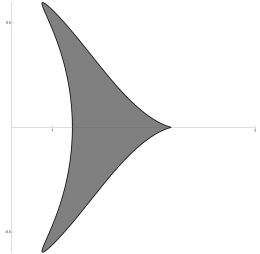
	Ω Bounded	Ω Unbounded
$0 \notin \Omega$		
$0 \in \Omega$		

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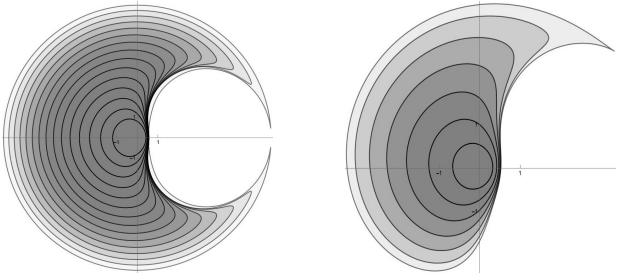
Theorem 2.12

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω not containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $c > 0$, $\lambda \in \mathbb{C}$, and $z_1 \in \mathbb{D}_*$ for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

$$\varphi(z) = cze^{\frac{\lambda}{1-z\overline{z_1}}}, \tag{4}$$

where $\varphi(z_1) = w_0$, and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1}\overline{\varphi'(z_1)}}$.

$w_0 = 1.9$
$\alpha = 1.5, 1.5 + .3i$



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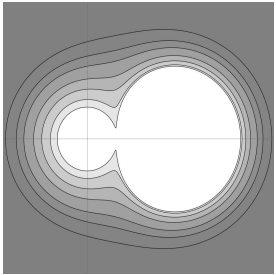
Theorem 2.13

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists a bounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0 \in \mathbb{D}$ for which $\Omega = \varphi(\mathbb{D})$, where φ is univalent and given by

$$\varphi(z) = \frac{w_0}{|z_0|} b_{z_0}(z) e^{\lambda z}, \tag{5}$$

where $\lambda = \frac{\overline{\alpha}}{\varphi'(0)}$.

$w_0 = 1$
$\alpha = .7$



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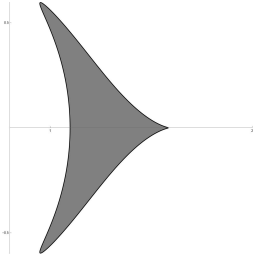
Theorem 2.14

Take $w_0, \alpha \in \mathbb{C} \setminus \{0\}$. There exists an unbounded simply connected domain Ω containing zero for which $\Omega \in QD_0\left(\frac{\alpha}{w-w_0}\right)$ if and only if there exist $\lambda \in \mathbb{C}$ and $z_0, z_1 \in \mathbb{D}_*$, for which $\Omega = \varphi(\mathbb{D}_*)$, where φ is univalent and given by

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{\frac{\lambda}{1-z\overline{z_1}}}, \tag{6}$$

where $\varphi(z_1) = w_0$ and $\lambda = \frac{\overline{\alpha w_0}}{\overline{z_1}\overline{\varphi'(z_1)}}$.

$w_0 = 2,$
$\alpha = -.15$



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- Generalize beyond simply connected domains.

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- Generalize beyond simply connected domains.
  - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with.

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  - Schottky-Klein Prime functions.

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- Analyze case in which $\partial\Omega$ contains a singular point.

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- Generalize fact that the boundary of a QD is an algebraic curve to LQDs.

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- Consider quadrature domains with "Abelian" quadrature functions.

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- Consider quadrature domains with "Abelian" quadrature functions.
- Characterize topology of generalized QDs?
  - Classical case: Lee & Makarov (2016)

That's all!

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Thank you!

The Faber Transform Method for Unbounded LQDs

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Theorem 3.1

Let $\Omega \in QD_0(h)$ be an unbounded simply connected domain with Riemann map φ. Then there exists $r \in \mathcal{A}(\mathbb{D})$ such that

$$\begin{aligned} \varphi(z) &= cze^{r^{\#}(z)}, && \text{when } 0 \notin \Omega, \\ \varphi(z) &= c|z_0|zb_{z_0}(z)e^{r^{\#}(z)}, && \text{when } 0 \in \Omega, \end{aligned} \tag{7}$$

Where $\varphi(z_0) = 0$.

The Faber Transform Method for Unbounded LQDs

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Where $\varphi(z_0) = 0$. In this case,

$$r(z) = \Phi_{\varphi}^{-1}\left(wh(w)\right)(z) - \Phi_{\varphi}^{-1}\left(wh(w)\right)(0). \tag{8}$$

and

$$h(w) = \frac{\Phi_{\varphi}\left(r\right)(w)}{w} \tag{9}$$

Symmetric Two Point LQDs

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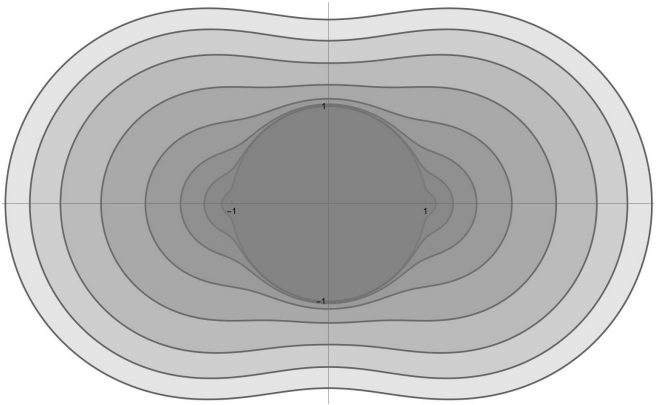

Family of LQDs in $\mathrm{QD}_0\left(\frac{\alpha}{w-1}+\frac{\alpha}{w+1}\right)$, for $0<\alpha<1$.

$$\varphi(z) = cze^{\frac{2z_+\alpha}{\varphi'(z_+)}\frac{z^2}{1-z^2z_+^2}}$$

Singular Monomial LQDs

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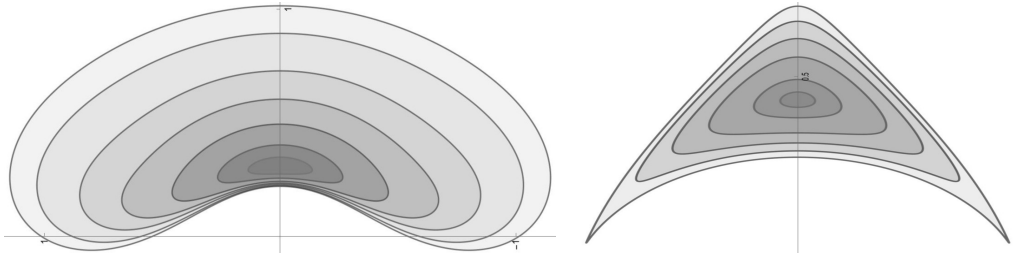

Family of complements of singular monomial LQDs in $\mathrm{QD}_0(2\alpha w)$. For $\alpha = -2$, $q = -2$, $0 < c \leq .3$ (left); and $\alpha = 1$, $q = -2$, $0 < c < .25$ (right). The plot is rotated so the positive real axis points upwards.

$$\varphi(z) = c|z_0| z b_{z_0}(z) e^{2\overline{\alpha}\left(c^2 z^{-2} + \beta z^{-1}\right)}.$$

Monomial LQDs

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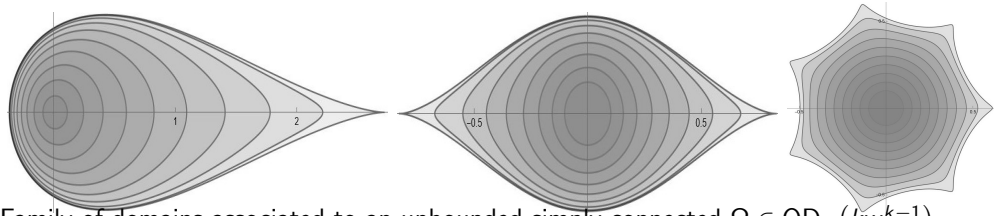
If $0 \notin \Omega \in \mathrm{QD}_0(\alpha k w^{k-1})$ is simply connected, then $\varphi(z) = cze^{\overline{\alpha} k c^k z^{-k}}$ is a Riemann map for Ω. (c is the conformal radius of Ω)



Family of domains associated to an unbounded simply connected $\Omega \in \mathrm{QD}_0\left(kw^{k-1}\right)$, $k \in \{1, 2, 7\}$.

Transcendental Antiholomorphic Dynamics

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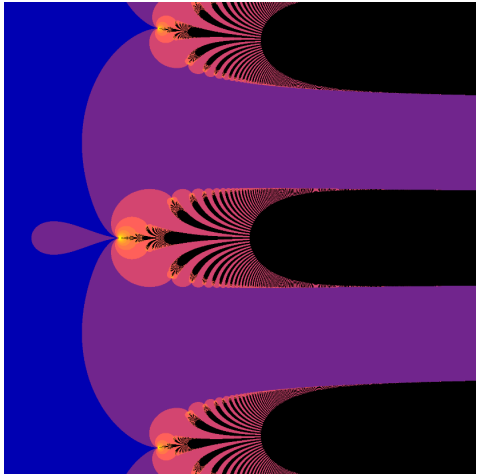

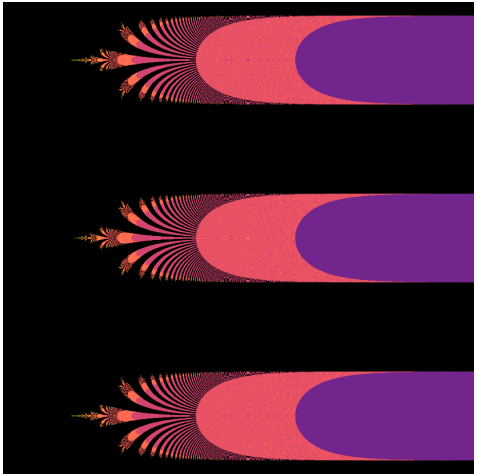

Tiling set of the Schwarz reflection σ associated to $\Omega \in \mathrm{QD}_0(1)$ (left) and filled Julia set of the anti-holomorphic exponential map $w \mapsto e^{\overline{w}-1}$ (right).

Power-Weighted Quadrature Domains

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Definition 3.2 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)} dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

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$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

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We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

When $n = 1$, we recover classical QDs. On the other hand, when $n > 1$ then the metric is singular at 0. This results in interesting behaviour.

Power-Weighted Quadrature Domains

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Definition 3.2 (Power-Weighted Quadrature Domain)

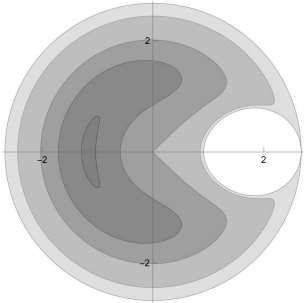
We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)} dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

When $n = 1$, we recover classical QDs. On the other hand, when $n > 1$ then the metric is singular at 0. This results in interesting behaviour.

For example when $\Omega \in \mathrm{QD}_2\left(\frac{10}{w-2}\right)$, a corner appears when $\partial\Omega$ intersects the origin.



Power-Weighted Faber Transform Identity

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The Faber transform formulae for classical quadrature domains generalize nicely to weighted QDs.

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The Faber transform formulae for classical quadrature domains generalize nicely to weighted QDs.
If $n \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_n(h)$ is bounded and s.c, then

$$\varphi^n(z) = \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{h(w) + G(w)}{nw^{n-1}}\right)^{\#}(z),$$

for some $G \in \mathcal{A}(\Omega)$.

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If $n \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_n(h)$ is bounded and s.c, then

$$\varphi^n(z) = \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{h(w) + G(w)}{nw^{n-1}}\right)^{\#}(z),$$

for some $G \in \mathcal{A}(\Omega)$.

Similarly, if $\Omega \in \mathrm{QD}_n(h)$ is unbounded, then

$$\varphi^n(z) = W_n(z) - W_n(0) + \Phi_\varphi^{-1}\left(\frac{h + G(w)}{R'}\right)^{\#}(z)$$

for some $G \in \mathcal{A}_0(\Omega)$, where $W_n := \Phi_\varphi^{-1}(z^n)$ is the nth "inverse Faber polynomial".

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If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

[8] Similar result in [Dragnev, Legg & Saff (2022)]

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$$\begin{aligned}\varphi^n(z) &= \varphi^n(0) + \Phi_\varphi^{-1}\left(\frac{c}{nw^{n-1}(w-w_0)}\right)^{\#}(z) \\ &= w_0^n + \frac{c}{n\overline{w_0}^{n-1}}\Phi_\varphi^{-1}\left(\frac{1}{w-w_0} + O(1)\right)^{\#}(z)\end{aligned}$$

[8] Similar result in [Dragnev, Legg & Saff (2022)]

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[8] Similar result in [Dragnev, Legg & Saff (2022)]

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So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$.

[8] Similar result in [Dragnev, Legg & Saff (2022)]

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So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$. Because $1 \in \mathcal{A}(\Omega)$,

$$c = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{c}{w-w_0}dw = \int_\Omega n^2|w|^{2(n-1)}dA(w) = \alpha^2.$$

So $\alpha = \sqrt{c}$.

[8] Similar result in [Dragnev, Legg & Saff (2022)]

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So Ω is an nth root of $\mathbb{D}_\alpha(w_0^n)$. Because $1 \in \mathcal{A}(\Omega)$,

$$c = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{c}{w-w_0}dw = \int_\Omega n^2|w|^{2(n-1)}dA(w) = \alpha^2.$$

So $\alpha = \sqrt{c}$. $\longrightarrow$ $\Omega = \left(\mathbb{D}_{\sqrt{c}}(w_0^n)\right)^{\frac{1}{n}}$ (taking the principal branch of $(\cdot)^{\frac{1}{n}}$).[8]

[8] Similar result in [Dragnev, Legg & Saff (2022)]

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Lemma 3.3

Let $n \in \mathbb{Z}_+$, $\Omega \ni w_0$ be a bounded, simply connected domain admitting a Riemann map $\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = w_0$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then,

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = w_0 (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}} .$$

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$$\varphi(z) = w_0 (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}} .$$
2. *If $w_0 = 0$ (so $0 \in \Omega$), then there exists $h \in X_k(0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_{k-1} \in \mathbb{D} \setminus \{0\}$ and $a > 0$ such that*
$$\varphi(z) = az (1 - zz_1)^{\frac{1}{n}} \ldots (1 - zz_{k-1})^{\frac{1}{n}} .$$

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3. *If $0 \in \Omega$ and $w_0 \neq 0$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
$$\varphi(z) = \frac{w_0}{|z_1|^2} \frac{z - z_1}{z - \overline{z_1}^{-1}} (1 - z\overline{z_1})^{\frac{1}{n}} (1 - zz_2)^{\frac{1}{n}} \ldots (1 - zz_k)^{\frac{1}{n}}.$$

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Lemma 3.4

Let $\Omega \ni w_0$ be an unbounded, simply connected domain with conformal radius a admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ with $\varphi\left(\overline{z_0}^{-1}\right) = w_0$, and $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0\}$ such that*

$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \cdots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

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1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exists $z_0 \in \mathbb{D} \setminus \{0\}$ with $\varphi\left(\overline{z_0}^{-1}\right) = w_0$, and $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0\}$ such that*
$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \cdots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

2. *If $0 \in \Omega$ then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_0, z_1 \in \mathbb{D} \setminus \{0\}$ with $\varphi(\overline{z_0}^{-1}) = w_0$, and $z_2, \ldots, z_k \in \mathbb{D} \setminus \{0, z_0, z_1\}$ such that*
$$\varphi(z) = az\left(1 - \frac{z_0}{z}\right)^{-\frac{k}{n}} \frac{z - \overline{z_1}^{-1}}{z - z_1}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \cdots \left(1 - \frac{z_k}{z}\right)^{\frac{1}{n}}.$$

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Lemma 3.5

Let Ω be an unbounded, simply connected domain with conformal radius a, admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$. Then for each $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_{\geq 0}$,

1. *If $0 \notin \Omega$, then there exists a polynomial h of degree k such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_{k+1} \in \mathbb{D} \setminus \{0\}$ such that*

$$\varphi(z) = az\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \cdots \left(1 - \frac{z_{k+1}}{z}\right)^{\frac{1}{n}}.$$

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$$\varphi(z) = az\frac{z - \overline{z_1}^{-1}}{z - z_1}\left(1 - \frac{z_1}{z}\right)^{\frac{1}{n}} \cdots \left(1 - \frac{z_{k+1}}{z}\right)^{\frac{1}{n}}.$$